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Variational Principles for Heat Transfer

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1. TERMINOLOGY

A variational principle is based on a functional, which is a correspondence assigning a real number to each function in a given class of functions. The functional is made stationary (preferably, but not always, a minimum) with respect to changes or variations in the function. This terminology agrees with the classical development (see [1,2,3]). In this paper, so-called variational principles which do not have a functional, or for which the functional is not stationary, are called quasi-variational principles or restricted variational principles.

This distinction is similar to the distinction between d'Alembert's principle and Hamilton's principle for the movement of a system of particles. D'Alembert's principle is

$$\widehat{\delta W} = \sum_{k=1}^{N} (\underset{\sim}{F}_k - m_k \underset{\sim}{A}_k) \cdot \delta \underset{\sim}{R}_k = 0 \tag{1}$$

where $\underset{\sim}{F}_k$ is the net force on the k-th particle, m_k and $\underset{\sim}{A}_k$ are the mass and acceleration of the k-th particle. The virtual work $\widehat{\delta W}$ is a differential form, i.e., there is no W whose variation gives $\widehat{\delta W}$. The $\delta \underset{\sim}{R}_k$ is an infinitesimal displacement. Hamilton's principle is obtained by integrating over time.

$$A \equiv \int_{t_1}^{t_2} L dt, \quad L \equiv T - V. \tag{2}$$

$$\int_{t_1}^{t_2} \widehat{\delta W} dt = \delta \int_{t_1}^{t_2} (T - V) dt. \tag{3}$$

Here T is the kinetic energy, V is the potential energy, L is the Lagrangian and A is the action integral. Such a formulation is possible if the forces are derivable from a potential. Note that in Hamilton's principle a functional, A, exists and is made stationary. Hamilton's principle is thus a true variational principle whereas d'Alembert's principle is a quasi-variational principle.

2. FRÉCHET DERIVATIVES

The scientist and engineer usually have a differential or integral equation and the question arises whether or not a variational principle exists for that equation. Fréchet derivatives are used to answer this question [4]. Consider the differential equation, possibly nonlinear

$$N(u) = 0. \tag{4}$$

The Fréchet differential of the operator N in the direction ϕ is

$$N'_u \phi \equiv \lim_{\varepsilon \to 0} \frac{N(u+\varepsilon\phi) - N(u)}{\varepsilon} = \frac{\partial}{\partial \varepsilon} [N(u+\varepsilon\phi)] \bigg|_{\varepsilon = 0} \tag{5}$$

N'_u is the Fréchet derivative of the operator N. A variational principle exists if the operator N'_u is symmetric.

$$\int \psi\, N'_u\, \phi\, dV = \int \phi N'_u\, \psi dV. \tag{6}$$

This condition is applied below to answer the question of whether a variational principle exists for various forms of the heat transfer equations. The application is described in detail elsewhere [4].

3. VARIATIONAL PRINCIPLES

3.1 Steady-State, Linear Heat Conduction

The equations for temperature, T, are

$$\nabla \cdot (k \nabla T) = f(\underset{\sim}{x}) \text{ in } V, \tag{7}$$

$$T = T_1\ (\underset{\sim}{x}) \text{ on } S_1, \tag{8}$$

$$-k\underset{\sim}{n} \cdot \nabla T = q_2(x) \text{ on } S_2, \tag{9}$$

$$-k\underset{\sim}{n} \cdot \nabla T = h(T - T_3(\underset{\sim}{x})) \text{ on } S_3, \tag{10}$$

where the thermal conductivity, k, and heat transfer coefficient, h, are functions of position, but not temperature. The functions, T_1, q_2, and T_3 are specified on their respective boundaries, which may be null. The variational principle is

$$\Phi(T) = \int_V [1/2k\nabla T\cdot\nabla T + Tf(\underset{\sim}{x})]\, dV + \int_{S_2} q_2 T dS + 1/2 \int_{S_3} h(T-T_3)^2 dS, \qquad (11)$$

and the function Φ is to be made stationary among functions T satisfying $T = T_1$ on S_1 and which are continuous with piecewise continuous first derivatives. Note that for each function T there is a real number Φ, making Φ a functional, and the variations of Φ with respect to T lead to Eq. (7) as Euler equation and Eq. (9-10) as natural boundary conditions. The variations give:

$$\delta\Phi = \left.\frac{d\Phi(T+\varepsilon\delta T)}{d\varepsilon}\right|_{\varepsilon=0} =$$

$$\int_V [k\nabla T\cdot\nabla\delta T + \delta T\, f(\underset{\sim}{x})]dV + \int_{S_2} q_2\, \delta T dS + \int_{S_3} h\,(T-T_3)\delta T dS. \qquad (12)$$

Using the divergence theorem and setting $\delta\Phi = 0$ gives

$$\int_V \delta T\, [-\nabla\cdot(k\nabla T) + f(\underset{\sim}{x})]dV + \int_{S_2} \delta T\, [q_2 + k\underset{\sim}{n}\cdot\nabla T]dS + \int_{S_3} \delta T[h(T-T_3) + k\underset{\sim}{n}\cdot\nabla T]dS + \int_{S_1} \delta T\, k\underset{\sim}{n}\cdot\nabla T dS = 0. \qquad (13)$$

Since $\delta T = 0$ on S_1 (the trial function must satisfy $T = T_1$ on S_1) the last integral vanishes. The remaining integrands in Eq. (13) are the desired differential equation (7), and natural boundary conditions (9-10).

3.2 Steady-State, Nonlinear Heat Conduction

When k and h depend on temperature, we make the transformation

$$\phi = \int_{T_o}^{T} k(\xi)d\xi, \quad \nabla\phi = k\nabla T. \tag{14}$$

This leads to equations of the form Eq. (7-10) except that Eq. (10) becomes

$$-\hat{n}\cdot\nabla\phi = h(\phi)[g(\phi) - g(\phi_3)] \text{ on } S_3. \tag{15}$$

$g(\phi)$ is the inverse transformation of Eq. (14). The boundary term on S_3 in Eq. (11) is then

$$\int_{S_3}\int_{\phi_3}^{\phi} h(\xi)[g(\xi) - g(\phi_3)]\, d\xi dS. \tag{16}$$

Thus nonlinear functions k(T) and/or h(T) can be handled.

If the boundary condition is a radiation condition

$$-k(\hat{x})\hat{n}\cdot\nabla T = h(\hat{x})(T^n - T_3^n), \tag{17}$$

then the boundary term on S_3 is

$$\int_{S_3}\int_{T_3}^{T} h(\xi^n - T_3^n)d\xi dS = \int_{S_3} [\frac{h}{n+1}(T^{n+1} - T_3^{n+1}) - hT_3^n(T-T_3)]dS. \tag{18}$$

Thus, simple radiation boundary conditions can be handled with variational principles.

If the heat generation term is nonlinear, as in reaction-diffusion problems or combustion problems, a variational principle also exists. For example, consider the following equation.

$$\nabla^2 T = e^T. \tag{19}$$

By letting $N(T) = \nabla^2 T - e^T$ we obtain the Fréchet differential from Eq. (5).

$$N_T' \phi = \nabla^2\phi - e^T\phi. \tag{20}$$

Equation (6) then becomes

$$\int [\psi\nabla^2\phi - e^T\psi\phi]dV \overset{?}{=} \int [\phi\nabla^2\psi - e^T\phi\psi]dV. \qquad (21)$$

The divergence theorem can be used to show that these are the same under appropriate boundary conditions. For the more general equation,

$$\nabla\cdot(k\nabla T) = f(\underset{\sim}{x},T), \qquad (22)$$

the variational integral is an extension of Eq. (11), with the term Tf(x) replaced by

$$\int_{T_o}^{T} f(\underset{\sim}{x},\xi)d\xi, \qquad (23)$$

with T_o an arbitrary reference temperature.

The Euler equation comes from

$$\int_V \delta T\ [\nabla\cdot(k\nabla T) - f(\underset{\sim}{x},T)]dV = 0. \qquad (24)$$

3.3 Steady-State, Linear Heat Convection

The equation for combined heat conduction and convection is

$$\hat{u}\cdot\nabla T = \alpha\nabla^2 T, \qquad (25)$$

where $\hat{u}$ is a known velocity field and α is the thermal diffusivity. Fréchet derivatives show that no variational principle exists for Eq. (25). Equation (6) applied to the troublesome convection term is

$$\int \psi\underset{\sim}{u}\cdot\nabla\phi dV \neq \int \phi\underset{\sim}{u}\cdot\nabla\psi dV. \qquad (26)$$

The Fréchet derivative is not symmetric.

If an integrating factor is used,

$$g(T,\nabla T)[\hat{u}\cdot\nabla T - \alpha\nabla^2 T] = 0, \qquad (27)$$

then Fréchet derivatives give a variational principle only if the velocity is derivable from a potential [4],

$$\hat{u} = -\nabla\Omega. \qquad (28)$$

The integrating factor is $g = \exp(\Omega/\alpha)$ and the variational integral is

$$\Phi(T) = -\tfrac{1}{2} \int_V \exp(\Omega/\alpha) \nabla T \cdot \nabla T dV$$

$$-\tfrac{1}{2} \int \exp(\Omega/\alpha) \frac{h}{\rho Cp} (T-T_s)^2 dS. \quad (29)$$

Since velocity fields are given by Eq. (28) only in unusual cases, a variational principle seldom applies to heat convection. Even this transformation does not work if the thermal conductivity depends on temperature [4].

3.4 Unsteady-State, Linear Heat Conduction

Another complication is the unsteady-state problem:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T, \quad (30)$$

$$T = T_o \text{ at } t = 0, \quad (31)$$

$$T = T_1 \text{ on } S_1. \quad (32)$$

Fréchet derivatives show that a variational principle does not exist for the equation in this form [4]. The Laplace transform can be taken, and then a variational principle exists.

With $$\bar{T} \equiv L[T] \quad (33)$$

the Laplace transform of Eq. (30-31) is

$$s\bar{T} - T_o = \alpha \nabla^2 \bar{T}, \quad (34)$$

$$\bar{T} = \bar{T}_1 \text{ on } S_1. \quad (35)$$

We divide by s and provide a variational functional

$$\Phi(\bar{T}) = \int_V [\frac{1}{2} \frac{\alpha}{s} \nabla\bar{T} \cdot \nabla\bar{T} + \frac{1}{2} \bar{T}^2 - \frac{1}{s} \bar{T}\, T_o] dV. \quad (36)$$

The variation gives

$$\delta\Phi = \int_V \delta\bar{T} \ [- \frac{\alpha}{s} \nabla^2\bar{T} + \bar{T} - \frac{1}{s} T_o] dV = 0. \quad (37)$$

Gurtin [5] used convolution integrals to provide a variational principle:

$$\Phi(T;t) = \tfrac{1}{2} \int_V [T*T + \alpha*\nabla T*\nabla T - 2T_o*T] dV. \quad (38)$$

The convolution is defined as

$$\int_V u*v\ dV = \int_V \int_0^t u(t-\tau,\underset{\wedge}{x})v(\tau,\underset{\wedge}{x})d\tau dV, \tag{39}$$

and the variation of Eq. (38) gives the Euler equation

$$T-T_o = \int_0^t \alpha\nabla^2 T(\tau,\underset{\wedge}{x})d\tau \equiv \alpha*\nabla^2 T. \tag{40}$$

Eq. (40) is the integral version of Eq. (30).

Filippov and Skorokhodov [6] provide a variational integral involving spatial integrals rather than temporal integrals for one-dimensional problems:

$$\Phi(T) = \tfrac{1}{2}\int_o^t \int_a^b [\frac{1}{\alpha}\frac{\partial(T^2)}{\partial t} + (\frac{1}{\alpha}\int_0^x \frac{\partial T}{\partial t}(\xi,t)d\xi)^2 + (\frac{\partial T}{\partial x})^2]dxdt. \tag{41}$$

These last variational integrals, Eq. (38,41) are so specialized that their extensions to nonlinear problems are not at all clear.

Equation (30) has an adjoint problem, and the techniques to find it are described elsewhere [4].

$$-\frac{\partial T^*}{\partial t} = \alpha\nabla^2 T^*, \tag{42}$$

$$T^* = T_o \text{ at } t = t_f, \tag{43}$$

$$T^* = T_1^* \text{ on } S_1. \tag{44}$$

The "final" time is t_f. A variational principle for the combined problem, Eq. (30-32,42-44) was given by Morse and Feshback [7].

$$\Phi[T,T^*] = \int_0^{t_f}\int_V [\alpha\nabla T\cdot\nabla T^* + \frac{1}{2}(T^*\frac{\partial T}{\partial t} - T\frac{\partial T^*}{\partial t})]dV$$

$$+ \frac{1}{2}\int [T_o(T^* - T)]_o^{t_f}\ dV. \tag{45}$$

Taking variations with respect to T gives the following equation.

$$\delta_T \Phi = \int_0^{t_f} \int_V \delta T \, [-\alpha\nabla^2 T* - \frac{\partial T*}{\partial t}] dV + \frac{1}{2} \int_V \delta T \, [T* - T_o] \Big|_o^{t_f} dV. \qquad (46)$$

Since $\delta T = 0$ at $t = 0$ and $T* = T_o$ at $t = t_f$ we get Eq. (42). Taking variations with respect to T* gives another equation.

$$\delta_{T*} \Phi = \int_0^{t_f} \int_V \delta T* \, [-\alpha\nabla^2 T + \frac{\partial T}{\partial t}] dV + \frac{1}{2} \int_V \delta T* \, [T_o - T] \Big|_o^{t_f} dV. \qquad (47)$$

Since $\delta T* = 0$ at $t = t_f$ and $T = T_o$ at $t = 0$ we get the Euler equation (30).

3.5 Applications to Unsteady-State Heat Conduction

We have thus provided four variational principles for the unsteady-state heat conduction equation (30). In application these can be shown to give identical results. When we apply Laplace transforms, Eq. (36), we expand in a trial function

$$\overline{T}^N = \frac{1}{s} T_1 + \sum_{j=1}^{N} \overline{a}_j(s) T_j(\underset{\sim}{x}). \qquad (48)$$

(We consider only the case of T_1 a constant, for simplicity.) Eq. (37) becomes

$$\int_V T_i(\underset{\sim}{x}) \, [- \frac{\alpha}{s} \nabla^2 \overline{T}^N + \overline{T}^N - \frac{1}{s} T_o] dV = 0. \qquad (49)$$

Since

$$L^{-1} \, [\overline{u} \, \overline{v}] = \int_o^t u(t-\tau) \, v(\tau) \, d\tau \qquad (50)$$

we can take the inverse transform of Eq. (49) to get

$$\int_V T_i(\underset{\sim}{x}) \, [-\alpha \int_o^t 1 \cdot \nabla^2 T(\underset{\sim}{x},\tau) d\tau] dV + \int_V T_i(\underset{\sim}{x}) \, [T^N(\underset{\sim}{x},t) - T_o] dV = 0, \qquad (51)$$

Differentiating this once with respect to time gives

$$\int_V T_i(\underset{\sim}{x}) \left[\frac{\partial T^N}{\partial t} (\underset{\sim}{x},t) - \alpha \nabla^2 T^N(\underset{\sim}{x},t)\right] dV = 0. \quad (52)$$

This is identical to the Galerkin method applied to the same problem (30).

When we apply convolution integrals in Eq. (38), we use the trial function

$$T^N = T_1 + \sum_{j=1}^{N} a_j(t)\, T_j(x). \quad (53)$$

The variation of Eq. (38) gives

$$\delta\Phi = \int_V \delta T^* \left[T - \alpha * \nabla^2 T - T_o\right] dV = 0. \quad (54)$$

Using Eq. (53) gives

$$\delta\Phi = \int_V T_j(x) * \left[T^N - \alpha * \nabla^2 T^N - T_o\right] dV = 0 \quad (55)$$

$$= \int_V \int_o^t T_j(\underset{\sim}{x}) \left[T^N(\underset{\sim}{x},\tau) - \int_o^{\tau} \alpha \nabla^2 T^N(\underset{\sim}{x},\xi) d\xi - T_o(\underset{\sim}{x})\right] dV. \quad (56)$$

Differentiation of this twice with respect to t gives Eq. (52).

Next we apply the variational principle (41). This case is restricted to one dimension, $a \le x \le b$. The first variation gives

$$\delta\Phi = \int_o^t \int_a^b \left[\frac{1}{\alpha} \frac{\partial}{\partial t} (T\delta T) + \frac{1}{\alpha^2} \int_o^x \frac{\partial T}{\partial t} (\xi,t) d\xi \int_o^x \frac{\partial \delta T}{\partial t} (\xi,t) d\xi + \frac{\partial T}{\partial x} \frac{\partial \delta T}{\partial x}\right] dx dt. \quad (57)$$

Henceforth we use $w(x,t) \equiv \delta T(x,t)$. To obtain the correct Euler equations we first prove some identities. The identification B.T. denotes terms which can be evaluated on the boundary using the divergence theorem after integration over x and/or t.

$$f(\theta,t) \equiv \int_a^{\theta} \frac{\partial w}{\partial t} (\xi,t) d\xi \quad , \quad \frac{\partial f(x,t)}{\partial x} = \frac{\partial w}{\partial t} (x,t). \quad (58)$$

The following term is integrated by parts several times.

$$\int_a^b \frac{\partial^2 T}{\partial x^2} \int_b^x f(\theta,t)d\theta \; dx = - \int_a^b \frac{\partial T}{\partial x} f(x,t)dx + \text{B.T.}$$

$$= \int_a^b T \frac{\partial f}{\partial x} dx + \text{B.T.} = \int_a^b T \frac{\partial w}{\partial t} dx. \qquad (59)$$

Next define a function h:

$$\frac{\partial h}{\partial x} = \frac{\partial T}{\partial t}(x,t) \; ; \quad h(x,t) = \int_a^x \frac{\partial T}{\partial t}(\xi,t)d\xi. \qquad (60)$$

Then the following integral can be integrated by parts several times.

$$\int_a^b \frac{\partial T}{\partial t}(x,t) \int_b^x f(\theta,t)d\theta dx = \int_a^b \frac{\partial h}{\partial x} \int_b^x f(\theta,t)d\theta dx \qquad (61)$$

$$= \int_a^b \frac{\partial}{\partial x}\left[h \int_b^x f(\theta,t)d\theta\right]dx - \int_a^b h \; f(x,t)dx \qquad (62)$$

$$= - \int_a^b \int_a^x \frac{\partial T}{\partial t}(\xi,t)d\xi \int_a^x \frac{\partial w}{\partial t}(\xi,t)d\xi \; dx + \text{B.T.} \qquad (63)$$

Finally we use

$$\int_o^t \frac{\partial}{\partial t}(Tw)dt = \int_o^t \frac{\partial T}{\partial t} w \; dt + \int_o^t T \frac{\partial w}{\partial t} dt \qquad (64)$$

and

$$\int_a^b \frac{\partial T}{\partial x}\frac{\partial w}{\partial x} dx = - \int_a^b w \frac{\partial^2 T}{\partial x^2} dx + \text{B.T.} \qquad (65)$$

We now put these into Eq. (57). The first term uses Eq. (64) and Eq. (59). The second term uses Eq. (63) while the third term uses Eq. (65).

$$\delta\Phi = \int_o^t \int_a^b \left[\frac{1}{\alpha}\frac{\partial T}{\partial t} w + \frac{1}{\alpha}\frac{\partial^2 T}{\partial x^2} \int_b^x f(\theta,t)d\theta \right.$$

$$\left. - \frac{1}{\alpha^2}\frac{\partial T}{\partial t}(x,t) \int_b^x f(\theta,t)d\theta - w\frac{\partial^2 T}{\partial x^2}\right]dxdt + B.T. \qquad (66)$$

Rearrangement of Eq. (66) gives the Euler equation.

$$0 = \delta\Phi = \int_o^t \int_a^b \left[\frac{1}{\alpha}\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2}\right] \left[w - \frac{1}{\alpha}\int_b^x f(\theta,t)d\theta\right]dxdt \qquad (67)$$

$$= \int_o^t \int_a^b \left[\frac{1}{\alpha}\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2}\right] \left[w - \frac{1}{\alpha}\int_b^x \int_a^\theta \frac{\partial w}{\partial t}(\xi,t)d\xi d\theta\right]dxdt. \qquad (68)$$

If we use an expansion of the type in Eq. (53), where $T_j(x)$ is known but $a_j(t)$ is not, then $w = T_j(x)$ and we get from Eq. (68)

$$\int_o^t \int_a^b \left[\frac{1}{\alpha}\frac{\partial T^N}{\partial t} - \frac{\partial^2 T^N}{\partial x^2}\right] T_j(x)\, dxdt = 0. \qquad (69)$$

This is also equivalent to a Galerkin method.

Finally, in applying the fourth variational principle, the adjoint variational principle, we expand T as in Eq. (53) and write a similar expansion for T*.

$$T^{*N} = T_I^* + \sum_{j=1}^N a_j^*(t)\, T_j(x). \qquad (70)$$

Equation (47) then becomes

$$\int_0^{t_f} \int_V T_j(\underset{\sim}{x}) \left[-\alpha\nabla^2 T^N + \frac{\partial T^N}{\partial t}\right]dV = 0 \qquad (71)$$

or the same as Eq. (52). Thus all four variational principles lead to the same equations, and these are the same as applications of Galerkin's method.

4. VARIATIONS ON THE VARIATIONAL PRINCIPLES

Sometimes the search for a variational principle is fruitless. If a principle does not exist for the equation in one form, it may for another form of the same equation, such as Eq. (27), or Eq. (36, 38, 41, 45). If all these approaches fail a variational principle in the form of least-squares principle of the Method of Weighted Residuals always holds. For example, for Eq. (30) the variational integral can be taken as

$$\Phi(T) = \int_0^t \int_V \left[\frac{\partial T}{\partial t} - \alpha\nabla^2 T \right]^2 dVdt. \qquad (72)$$

The trial functions must be smooth since higher derivatives appear in Eq. (72) than Eq. (45), for example.

Reciprocal variational principles are sometimes useful for giving error bounds or special meaning to the variational integral. Minimum variational principles for eigenvalues can be used to give quite close upper and lower bounds on eigenvalues. A variational principle may lead to error bounds on the solution or proofs of uniqueness.

In applications the variational method will lead to symmetric matrices, and the linear algebra problem is more quickly solved than one with unsymmetric matrices. This is an important advantage over the Galerkin method. If there is a variational method applicable to a problem, the Galerkin method should be applied in a way that leads to equivalence with the variational method which is usually achieved by appropriate integration by parts. The variational principle also identifies the natural and essential boundary conditions. By contrast, the Galerkin method must be properly formulated to insure the boundary conditions are physically meaningful.

The heat conduction equation has led to a variety of quasi-variational principles and restricted variational principles. Rosen [8] and Glansdorff and Prigogine [local potential, 9] and Gyarmati [10] have constructed restricted variational principles--called restricted because certain variables are held constant during the variation but then allowed to be variable.

For Eq. (30) in one-dimension their restricted principles give

$$\Phi(T(x,t),\ T_o(x,t)) = \int_a^b \left[T \frac{\partial T_o}{\partial t} + \alpha\left(\frac{\partial T}{\partial x}\right)^2 \right]dx. \qquad (73)$$

Formal variation of T, keeping To fixed, and use of integration by parts, gives

$$\delta_T \Phi = \int_a^b \delta T \left(\frac{\partial T_o}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2}\right) dx + \text{B.T.} \tag{74}$$

This is clearly not the required equation, so now we set $T_o = T$, even though we could not do this in Eq. (73). If T_o were a given, known function, then Eq. (74) would make Φ stationary. Use of $T_o = T$, however, destroys the stationary character, as described by Finlayson and Scriven [11,4]. Thus the functional is not even stationary. The same formal operations can be used on any equation. The advantages attributed above to variational principles do not hold for such variational principles.

Biot [12] provides an alternative which is analogous to d'Alembert's principle in that no functional exists. For Eq. (30) he introduces a new heat-flow vector, defined such that

$$T = -\nabla \cdot \underset{\sim}{H}. \tag{75}$$

The quasi-variational principle is stated as

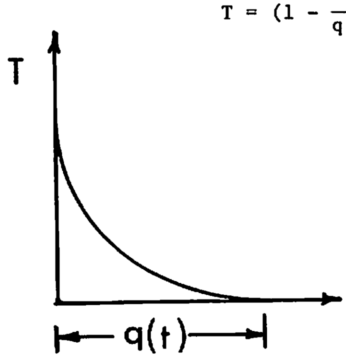
$$\int_V \delta \underset{\sim}{H} \cdot \left[\nabla T + \frac{1}{\alpha} \frac{\partial \underset{\sim}{H}}{\partial t}\right] dV = 0. \tag{76}$$

The Euler equation is

$$\nabla T + \frac{1}{\alpha} \frac{\partial \underset{\sim}{H}}{\partial t} = 0 \tag{77}$$

and if one takes the divergence of this equation and uses Eq. (75) one gets Eq. (30). Biot uses this quasi-variational principle in some innovative ways, as is also done in the heat integral method [13]. For heat conduction in a slab, when the wall temperature has suddenly jumped, a heat penetration distance is defined, and the approximate solution is defined over that distance.

$$T = \left(1 - \frac{x}{q(t)}\right)^2$$



Use of the quasi-variational principle gives an equation for q(t). Approaches such as these, which can be done with Galerkin, integral and other methods, are quite useful for engineering purposes.

Another attempt to obtain a variational principle is provided by Vujanovic [14]. He adds a term to Eq. (30).

$$m \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}. \qquad (79)$$

The variational integral is then

$$\Phi(T) = \frac{1}{2} \int_0^t \int_a^b [m (\frac{\partial T}{\partial t})^2 - \alpha (\frac{\partial T}{\partial x})^2] e^{t/m} dxdt. \qquad (80)$$

The first variation gives

$$\delta\Phi = \int_0^t \int_a^b \partial T [-m \frac{\partial^2 T}{\partial t^2} - \frac{\partial T}{\partial t} + \alpha \frac{\partial^2 T}{\partial x^2}] e^{t/m} dxdt + B.T. \qquad (81)$$

thus giving the right Euler equation, (79). Then Vujanovic sets m=0 to turn Eq. (79) into Eq. (30) and uses Eq. (81) with m→0. No consideration is given to the fact that this procedure creates a singular perturbation problem, the number of boundary conditions changes for a well-posed problem, and the solution T actually depends on m and no proof is given that the second and third integrals in Eq. (81) actually dominate the first integral (they do if the solution of Eq. (79) is independent of m). In any case, the variational integral (80) is undefined for m=0 and applications again are equivalent to Galerkin's method.

All these "principles" lead to methods which are identical to Galerkin methods, yet have no functional which is made stationary. They have not led to new insights. Their main impact is in the imaginative use of trial functions, such as Biot's treatment of heat conduction involving a heat penetration distance. Usually they are introduced and used to solve simple problems, in contrast to the Galerkin method which has found widespread use in the past decade.

5. CONCLUSIONS

Variational principles exist for some, but not all, heat transfer problems. The important ones are given and compared to quasi-variational principles and restricted variational principles. In applications Galerkin methods are often equivalent, and are certainly preferred if no variational principle exists.

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